Journal of Computational and Applied Mathematics 233 (2010) 1915–1926

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Existence of solutions of boundary value problems with integral boundary conditions for second-order impulsive integro-differential equations in Banach spaces ☆

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ARTICLE INFO

Article history:
Received 19 August 2007
Received in revised form 23 July 2009

MSC:
34B15

Keywords:
Boundary value problem
Integral boundary conditions
Fixed point theory
Existence
Measure of noncompactness

ABSTRACT

This paper investigates the existence of solutions for a class of second-order boundary-value problems with integral boundary conditions of nonlinear impulsive integro-differential equations in Banach spaces. The arguments are based upon the fixed point theorem of strict set contraction operators. Meanwhile, an example is worked out to demonstrate the main results.

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. The theory of impulsive differential equations has become an important area of investigation in recent years and is much richer than the corresponding theory of differential equations. For an introduction of the basic theory of impulsive differential equations in R^n, see [1–3] and the references therein, in the Banach space, see chapter 4 in [4].

On the other hand, the theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. For boundary-value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers in Gallardo [5–7] and the references therein. For more information about the general theory

☆ This work is sponsored by the National Natural Science Foundation of China (10671012), the Scientific Creative Platform Foundation of Beijing Municipal Commission of Education (PXM2008-014224-067420), the Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (PHR201008430) and the Scientific Research Common Program of Beijing Municipal Commission of Education (No-KM2010).

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doi:10.1016/j.cam.2009.07.060

of integral equations and their relation with boundary-value problems, we refer to the book of Corduneanu [8] and Agarwal and O'Regan [9].

Moreover, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary-value problems as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attentions. To identify a few, we refer the reader to [6,10–39,42] and references therein.

However, the corresponding theory for multipoint boundary-value problems for impulsive integro-differential equations in Banach spaces, especially in the case that the boundary value problem with integral boundary conditions in Banach spaces, is not investigated till now. Now, in this paper, we shall use fixed point theory in a cone for strict set contraction operators to investigate the existence of solutions for a class of second-order nonlinear impulsive integro-differential equations in a Banach space.

Consider the following boundary value problem with integral boundary conditions for second-order nonlinear impulsive integro-differential equation of mixed type in a real Banach space E:

$$\begin{cases} x''(t) + w(t)f(t, x(t), x'(t), (Ax)(t), (Bx)(t)) = \theta, \quad t \in J,\ t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) = x(1) = \displaystyle\int_0^1 v(s)x(s)\mathrm{d}s, \end{cases} \tag{1.1}$$

where $w \in C(J, [0, +\infty)), f \in C(J \times E \times E \times E \times E, E), J = [0, 1], 0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < 1, I_k \in C[E, E], \bar{I}_k \in C[E \times E, E]$, θ is the zero element of E, $v \in L^1[0, 1]$ is nonnegative, and

$$(Ax)(t) = \int_0^t g(t, s)x(s)\mathrm{d}s, \qquad (Bx)(t) = \int_0^1 h(t, s)x(s)\mathrm{d}s,$$

where $g \in C[D, R^+], D = \{(t, s) \in J \times J : t \geq s\}, h \in C[J \times J, R^+]$, R^+ is the set of all nonnegative real numbers, and $g_0 = \max\{g(t, s) : (t, s) \in D\}, h_0 = \max\{h(t, s) : (t, s) \in J \times J\}$. $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e.

$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-),$$

where $x(t_k^+)$ and $x(t_k^-)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t = t_k$ respectively. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$.

Let $PC[J, E] = \{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $x(t_k^+)$ exist for $k = 1, 2, \dots, m\}$, $PC^1[J, E] = \{x \in PC[J, E] : x'(t)$ exists at $t \neq t_k$ and is continuous at $t \neq t_k$, and $x'(t_k^+), x'(t_k^-)$ exist for $k = 1, 2, \dots, m\}$. $PC[J, E]$ is a Banach space with norm

$$\|x\|_{pc} = \sup_{t \in J} \|x(t)\|,$$

$PC^1[J, E]$ is a Banach space with norm

$$\|x\|_1 = \max\{\|x\|_{pc}, \|x'\|_{pc}\}.$$

Let $J' = J \setminus \{t_1, t_2, \dots, t_k, \dots, t_m\}$.

In scalar space, there are many authors who studied nonlinear multipoint boundary value problems. We refer readers to [10–21,29–39] and the references therein. Gupta and Trofimchuk in [16] concerned the existence of a solution for the following three-point boundary value problem

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) + e(t) = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = \alpha x(\eta), \end{cases} \tag{1.2}$$

where $0 < \eta < 1, \alpha \in R, \alpha > 1, \alpha\eta \neq 1$ are given, $f : [0, 1] \times R^2 \to R$ is a function satisfying Caratheodory's conditions and $e(t) \in L^1[0, 1]$. By using Leray–Schauder Continuation theorem, the authors obtained sharper existence conditions for the solvability of the above boundary value problem in a general case.

Recently, Ma and Castaneda in [19] studied the existence of positive solutions to the nonlinear boundary-value problem

$$\begin{cases} x''(t) + a(t)f(x(t)) = 0, \quad 0 < t < 1, \\ x'(0) = \displaystyle\sum_{i=1}^{n-2} b_i x'(\xi_i), x(1) = \sum_{i=1}^{n-2} a_i x(\xi_i), \end{cases} \tag{1.3}$$

under the following assumption:

(A_1) $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$. $a_i, b_i \in [0, +\infty)$ $(i = 1, 2, \dots, n-2)$ satisfying $0 < \sum_{i=1}^{n-2} a_i < 1$ and $\sum_{i=1}^{n-2} b_i < 1$.

(A_2) $f \in C([0,\infty),[0,\infty))$; $a \in C([0,1],[0,+\infty))$ and $a(t) \not\equiv 0$ on $[0,1]$.

The authors established the existence of at least one positive solution of (1.3) if f is either superlinear or sublinear by applying the fixed point theorem in a cone.

For abstract space, it is here worth mentioning that Guo and Lakshmikantham [22] discussed the multiple solutions of two-point boundary value problems of ordinary differential equations in Banach space. Guo in [23] obtained the sufficient condition for multiple positive solutions to a class of impulsive differential equations in abstract space. Guo in [24] investigated the existence of solutions to second-order impulsive differential equations in abstract space. In particular, we would like to mention some results of Guo [25–27]. In [25], Guo and Liu used fixed point index theory for cone mappings to investigate the existence of multiple positive solutions of a boundary-value problem for the following second-order impulsive differential equation:

$$\begin{cases} -x'' = f(t,x), \quad 0 \le t \le 1,\ t \ne t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad (k=1,2,\ldots,m), \\ ax(0) - bx'(0) = \theta, \qquad cx(1) + dx'(1) = \theta, \end{cases} \tag{1.4}$$

where $f \in C(J \times P, P)$, $J = [0,1]$, P is a cone in the real Banach space E, θ is the zero element of E, $f(t,\theta) = \theta$ for $t \in J$, $I_k \in C[P,P]$, $k = 1,2,\ldots,m$, $0 < t_1 < \cdots < t_i < \cdots < t_m < 1$, $a \ge 0$, $b \ge 0$, $c \ge 0$, $d \ge 0$ and $\delta = ac + ad + bc > 0$.

In [26], Guo considered the periodic boundary value problem for second-order impulsive integro-differential equations of mixed type in the Banach space E

$$\begin{cases} x'' + f(t,x,Tx,Sx) = \theta, \quad 0 \le t \le 2\pi,\ t \ne t_k, \\ \Delta x|_{t=t_k} = L_k x'(t_k), \\ \Delta x'|_{t=t_k} = L_k^* x(t_k), \quad (k=1,2,\ldots,m), \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{cases} \tag{1.5}$$

where $f \in C(J \times E \times E \times E, E)$, $J = [0,2\pi]$, $0 < t_1 < \cdots < t_i < \cdots < t_m < 2\pi$, L_k and L_k^*, $(k = 1,2,\ldots,m)$ are nonnegative constants. The author investigated the maximal and minimal solutions of periodic boundary value problem (1.5) by establishing a comparison result and using the upper and lower solutions.

Recently, Guo in [27] investigated the minimal nonnegative solution of the following initial value problem for a second-order nonlinear impulsive integro-differential equation of Volterra type on an infinite interval with an infinite number of impulsive times in a Banach space E:

$$\begin{cases} x'' = f(t,x,Tx), \quad \forall t \ge 0,\ t \ne t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), \quad (k=1,2,\ldots), \\ x(0) = x_0, \quad x'(0) = x_0^*, \end{cases} \tag{1.6}$$

where $f \in C(J \times P \times P \times P, E)$, $I_k, \bar{I}_k \in C[P,P]$, $J = [0,\infty)$, $x_0, x_0^* \in P$, $0 < t_1 < \cdots < t_k < \cdots < \cdots$, $t_k \to \infty$, as $k \to \infty$, P is a cone of E.

Very recently, Zhao and Chen in [32] studied the following m-point boundary value problem:

$$\begin{cases} x'' + a(t)f(t,x(t)) = \theta, \quad 0 < t < 1, \\ x'(0) = \displaystyle\sum_{i=1}^{m-2} b_i x'(\xi_i), \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases} \tag{1.7}$$

in Banach spaces E, where θ is zero element of E, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i, b_i \in [0,+\infty)$ $(i = 1,2,\ldots,m-2)$. By using the fixed point theorem of strict-set-contractions, they obtained some sufficient conditions for the existence of at least one or two positive solutions to problem (1.7).

Being directly inspired by [22,25,26], in the present paper, by using the fixed-point index theorem of strict-set-contractions, the authors prove some existence results of the problem (1.1) in the Banach space. The main features of this paper are as follows. Comparing with [4,16,19,22,25,26,29], we discuss the boundary value problem with integral boundary conditions, i.e., problem (1.1) includes second-order two, three, multipoint and nonlocal boundary value problems as special cases. To our knowledge, no paper considered second-order nonlinear impulsive integro-differential equation of mixed type with integral boundary conditions. Hence, we improve and generalize the results of [16,19,22,25,26,29] and some results in [4] to some degree, and so, it is interesting and important to study the existence of positive solutions for problem (1.1).

The organization of this paper is as follows. We shall introduce some lemmas and notations in the rest of this section. In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with BVP (1.1). In Section 3, the main result will be stated and proved. Finally, one example is also included to illustrate the main results.

Basic facts about ordered Banach space E can be found in [4,40,41]. Here, we just recall a few of them. The cone P in E induces a partial order on E, i.e., $x \le y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$ (N is called the normal constant of P). Without loss of generality, suppose, in present paper, the normal constant $N = 1$. If P is solid and $y - x \in \dot{P}$, we write $y \gg x$. For details on cone theory, see [40].

For a bounded set V in the Banach space E, we denote $\alpha(V)$ the Kuratowski measure of non-compactness (see [4,40,41], for further understanding). The operator $A : D \to E (D \subset E)$ is said to be a k-set contraction if $A : D \to E$ is continuous and bounded and there is a constant $k \ge 0$ such that $\alpha(A(S)) \le k\alpha(S)$ for any bounded $S \subset D$; a k-set contraction with $k < 1$ is called a strict set contraction.

In the following, $\alpha(\cdot)$, $\alpha_{PC^1}(\cdot)$ denote the Kuratowski's measure of noncompactness in E and $PC^1(J, E)$, respectively.

For the application in the sequel, we first state the following definition and lemmas which can be found in [4].

Definition 1.1. Let S be a bounded set of a real Banach space E. Let $\alpha(S) = \inf\{\delta > 0 : S$ can be expressed as the union of a finite number of sets such that the diameter of each set does not exeed δ, i.e., $S = \cup_{i=1}^m S_i$ with $\operatorname{diam}(S_i) \le \delta, i = 1, 2, \ldots, m\}$. Clearly, $0 \le \alpha(S) < \infty$. $\alpha(S)$ is called the Kuratowski's measure of noncompactness.

Lemma 1.1. *Let D be a bounded, closed and convex subset of the real Banach space E. If operator $A : D \longrightarrow D$ is a strict set contraction, then A has a fixed point in D.*

2. Preliminaries

To establish the existence of positive solutions in E of problem (1.1), let us list the following assumptions, which will stand throughout this paper.

(H_1) $f \in C(J \times E \times E \times E \times E, E)$, and for any $r > 0, f$ is uniformly continuous on $J \times B_r \times B_r \times B_r \times B_r$, where $B_r = \{x \in E : \|x\| \le r\}$. Further suppose that $\mu \in [0, 1)$, where $\mu = \int_0^1 v(s)\mathrm{d}s$.

(H_2) There exist nonnegative constants c_i, $i = 1, 2, 3, 4$, and d_k, $\bar{d}_k$, $\hat{d}_k$ such that

$$\alpha(f(t, u_1, u_2, u_3, u_4)) \le \sum_{i=1}^{4} c_i\alpha(u_i), \quad \forall t \in J, u_i \subset B_r \ (i = 1, 2, 3, 4), \tag{2.1}$$

$$\alpha(I_k(u_1)) \le d_k\alpha(u_1), \quad \forall u_1 \subset B_r \ (k = 1, 2, \ldots, m), \tag{2.2}$$

$$\alpha(\bar{I}_k(u_1, u_2)) \le \bar{d}_k\alpha(u_1) + \hat{d}_k\alpha(u_2), \quad \forall u_1, u_2 \subset B_r \ (k = 1, 2, \ldots, m), \tag{2.3}$$

and

$$\varGamma_r = \max\{\bar{\varGamma}_r, \hat{\varGamma}_r\} < 1, \tag{2.4}$$

where

$$\hat{\varGamma}_r = 2\max_{s\in J}\{w(s)\} \times (c_1 + c_2 + c_3g_0 + c_4h_0) + \sum_{k=1}^{m}[d_k + (2 - t_k)(\bar{d}_k + \hat{d}_k)],$$

$$\bar{\varGamma}_r = \frac{1}{2}\gamma \times \max_{s\in J}\{w(s)\} \times (c_1 + c_2 + c_3g_0 + c_4h_0) + \frac{3 - 2\mu}{1 - \mu}\sum_{k=1}^{m}[d_k + (1 - t_k)(\bar{d}_k + \hat{d}_k)],$$

here, γ is defined in (2.11).

(H_3) $w \in C(J, [0, +\infty))$, and there exists $t_0 \in J$ such that $w(t_0) > 0$.

We shall reduce problem (1.1) to an integral equation in E. To this end, we first consider operator T defined by

$$\begin{aligned}(Tx)(t) = &\int_0^1 H(t, s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s + \sum_{0<t_k<t}[I_k(x(t_k)) + (t - t_k)\bar{I}_k(x(t_k), x'(t_k))] \\ &- t\sum_{k=1}^{m}[I_k(x(t_k)) + (1 - t_k)\bar{I}_k(x(t_k), x'(t_k))] + \frac{1}{1 - \int_0^1 v(s)\mathrm{d}s}\int_0^1 \sum_{0<t_k<s}[I_k(x(t_k)) \\ &+ (s - t_k)\bar{I}_k(x(t_k), x'(t_k))]\mathrm{d}s - \frac{\int_0^1 sv(s)\mathrm{d}s}{1 - \int_0^1 v(s)\mathrm{d}s}\sum_{k=1}^{m}[I_k(x(t_k)) + (1 - t_k)\bar{I}_k(x(t_k), x'(t_k))],\end{aligned} \tag{2.5}$$

where

$$H(t, s) = G(t, s) + \frac{1}{1 - \mu}\int_0^1 G(s, \tau)v(\tau)\mathrm{d}\tau, \tag{2.6}$$

$$G(t, s) = \begin{cases} t(1 - s), & 0 \le t \le s \le 1, \\ s(1 - t), & 0 \le s \le t \le 1. \end{cases}$$

In what follows, we write $J_1 = [0, t_1], J_k = (t_{k-1}, t_k]$ $(k = 1, 2, \ldots, m)$. From (2.6) and the definition of $G(t, s)$, we can prove that $H(t, s)$, $G(t, s)$ have the following properties.

Proposition 2.1. *If $\mu \in [0, 1)$, then we have*

$$H(t,s) > 0, \qquad G(t,s) > 0, \quad \text{for } t, s \in (0,1), \tag{2.7}$$

$$H(t,s) \geq 0, \qquad G(t,s) \geq 0, \quad \text{for } t, s \in J. \tag{2.8}$$

Proposition 2.2. *For $t, s \in [0, 1]$, we have*

$$e(t)e(s) \leq G(t,s) \leq G(t,t) = t(1-t) = e(t) \leq \bar{e} = \max_{t\in[0,1]} e(t) = \frac{1}{4}. \tag{2.9}$$

Proposition 2.3. *If $\mu \in [0, 1)$, then for $t, s \in [0, 1]$, we have*

$$\rho e(s) \leq H(t,s) \leq \gamma s(1-s) = \gamma e(s) \leq \frac{1}{4}\gamma, \tag{2.10}$$

where

$$\gamma = \frac{1}{1-\mu}, \qquad \rho = \frac{\int_0^1 e(\tau)v(\tau)\mathrm{d}\tau}{1-\mu}. \tag{2.11}$$

Proof. By (2.6) and (2.8), we have

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1}{1-\mu}\int_0^1 G(s,\tau)v(\tau)\mathrm{d}\tau \\ &\geq \frac{1}{1-\mu}\int_0^1 G(s,\tau)v(\tau)\mathrm{d}\tau \\ &\geq \frac{\int_0^1 e(\tau)v(\tau)\mathrm{d}\tau}{1-\mu} s(1-s) \\ &= \rho e(s), \quad t \in [0,1]. \end{aligned} \tag{2.12}$$

On the other hand, noticing $G(t, s) \leq s(1 - s)$, we obtain

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1}{1-\mu}\int_0^1 G(s,\tau)v(\tau)\mathrm{d}\tau \\ &\leq s(1-s) + \frac{1}{1-\mu}\int_0^1 s(1-s)v(\tau)\mathrm{d}\tau \\ &\leq s(1-s)\left[1 + \frac{1}{1-\mu}\int_0^1 v(\tau)\mathrm{d}\tau\right] \\ &\leq s(1-s)\frac{1}{1-\mu} \\ &= \gamma e(s), \quad t \in [0,1]. \quad \diamond \end{aligned} \tag{2.13}$$

Lemma 2.1. *If $\mu \in [0, 1)$, then $x \in PC^1[J, E] \cap C^2[J', E]$ is a solution of problem (1.1) if and only if $x \in PC^1[J, E]$ is a solution of the following impulsive integral equation:*

$$\begin{aligned} x(t) = {} & \int_0^1 H(t,s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s + \sum_{0<t_k<t} [I_k(x(t_k)) + (t-t_k)\bar{I}_k(x(t_k), x'(t_k))] \\ & - t\sum_{k=1}^m [I_k(x(t_k)) + (1-t_k)\bar{I}_k(x(t_k), x'(t_k))] + \frac{1}{1-\int_0^1 v(s)\mathrm{d}s}\int_0^1 \left[\sum_{0<t_k<s} [I_k(x(t_k)) + (s-t_k)\bar{I}_k(x(t_k), x'(t_k))]\mathrm{d}s\right] \\ & - \frac{\int_0^1 sv(s)\mathrm{d}s}{1-\int_0^1 v(s)\mathrm{d}s}\sum_{k=1}^m [I_k(x(t_k)) + (1-t_k)\bar{I}_k(x(t_k), x'(t_k))], \end{aligned} \tag{2.14}$$

i.e., x is a fixed point of operator T defined by (2.5) in $PC^1[J, E]$.

Proof. First suppose that $x \in PC^1[J, E]$ is a solution of problem (1.1). It is easy to see by integration of (1.1) that

$$\begin{aligned} x'(t) &= x'(0) - \int_0^t w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s + \sum_{0<t_k<t} [x'(t_k^+) - x'(t_k)] \\ &= x'(0) - \int_0^t w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s + \sum_{0<t_k<t} \bar{I}_k(x(t_k), x'(t_k)). \end{aligned}$$

Integrating again, we can get

$$\begin{aligned} x(t) &= x(0) + x'(0)t - \int_0^t (t-s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)) + \sum_{0<t_k<t} \bar{I}_k(x(t_k), x'(t_k))(t - t_k). \end{aligned} \tag{2.15}$$

Letting $t = 1$ in (2.15), we find

$$\begin{aligned} x'(0) &= \int_0^1 (1-s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s \\ &\quad - \sum_{k=1}^m I_k(x(t_k)) - \sum_{k=1}^m \bar{I}_k(x(t_k), x'(t_k))(1 - t_k). \end{aligned} \tag{2.16}$$

Substituting $x(0) = \int_0^1 v(s)x(s)\mathrm{d}s$ and (2.16) into (2.15), we obtain

$$\begin{aligned} x(t) &= \int_0^1 v(s)x(s)\mathrm{d}s + t\left[\int_0^1 (1-s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s \right.\\ &\quad \left. - \sum_{k=1}^m I_k(x(t_k)) - \sum_{k=1}^m \bar{I}_k(x(t_k), x'(t_k))(1 - t_k)\right] - \int_0^t (t-s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)) + \sum_{0<t_k<t} \bar{I}_k(x(t_k), x'(t_k))(t - t_k) \\ &= \int_0^1 G(t,s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))\mathrm{d}s + t\left[-\sum_{k=1}^m I_k(x(t_k)) - \sum_{k=1}^m \bar{I}_k(x(t_k), x'(t_k))(1 - t_k)\right] \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)) + \sum_{0<t_k<t} \bar{I}_k(x(t_k), x'(t_k))(t - t_k) + \int_0^1 v(s)x(s)\mathrm{d}s, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \int_0^1 v(s)x(s)\mathrm{d}s &= \int_0^1 v(s)\left\{\int_0^1 G(s,u)w(u)f(u, x(u), x'(u), (Ax)(u), (Bx)(u))\mathrm{d}u - \left[\sum_{k=1}^m I_k(x(t_k))\right.\right. \\ &\quad \left.\left. + \sum_{k=1}^m \bar{I}_k(x(t_k), x'(t_k))(1 - t_k)\right] + \sum_{0<t_k<s} I_k(x(t_k)) + \sum_{0<t_k<s} \bar{I}_k(x(t_k), x'(t_k))(s - t_k) + \int_0^1 v(s)x(s)\mathrm{d}s\right\}\mathrm{d}s. \end{aligned} \tag{2.18}$$

Letting

$$\begin{aligned} \mathbb{A} &= \sum_{k=1}^m I_k(x(t_k)) + \sum_{k=1}^m \bar{I}_k(x(t_k), x'(t_k))(1 - t_k), \\ \mathbb{B} &= \sum_{0<t_k<s} I_k(x(t_k)) + \sum_{0<t_k<s} \bar{I}_k(x(t_k), x'(t_k))(s - t_k). \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 v(s)x(s)\mathrm{d}s &= \int_0^1 v(s)\left(\int_0^1 G(s,u)w(u)f(u, x(u), x'(u), (Ax)(u), (Bx)(u))\mathrm{d}u\right)\mathrm{d}s \\ &\quad - \mathbb{A}\int_0^1 v(s)s\mathrm{d}s + \int_0^1 \mathbb{B}v(s)\mathrm{d}s + \int_0^1 v(s)\mathrm{d}s \times \int_0^1 v(s)x(s)\mathrm{d}s, \end{aligned}$$

and so

$$\int_0^1 v(s)x(s)\mathrm{d}s = \frac{1}{1-\int_0^1 v(s)\mathrm{d}s}\left[\int_0^1 v(s)\left(\int_0^1 G(s,u)w(u)f(u,x(u),x'(u),(Ax)(u),(Bx)(u))\mathrm{d}u\right)\mathrm{d}s - \mathbb{A}\int_0^1 v(s)s\mathrm{d}s + \int_0^1 \mathbb{B}v(s)\mathrm{d}s\right]. \tag{2.19}$$

Substituting (2.19) into (2.17), we obtain

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)w(s)f(s,x(s),x'(s),(Ax)(s),(Bx)(s))\mathrm{d}s + t\left[-\sum_{k=1}^m I_k(x(t_k)) - \sum_{k=1}^m \bar{I}_k(x(t_k),x'(t_k))(1-t_k)\right] \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)) + \sum_{0<t_k<t}\bar{I}_k(x(t_k),x'(t_k))(t-t_k) \\ &\quad + \frac{1}{1-\int_0^1 v(s)\mathrm{d}s}\left[\int_0^1 v(s)\left(\int_0^1 G(s,u)w(u)f(u,x(u),x'(u),(Ax)(u),(Bx)(u))\mathrm{d}u\right)\mathrm{d}s\right. \\ &\quad \left. - \mathbb{A}\int_0^1 v(s)s\mathrm{d}s + \int_0^1 \mathbb{B}v(s)\mathrm{d}s\right] \\ &= \int_0^1 H(t,s)w(s)f(s,x(s),x'(s),(Ax)(s),(Bx)(s))\mathrm{d}s \\ &\quad + \sum_{0<t_k<t}[I_k(x(t_k)) + (t-t_k)\bar{I}_k(x(t_k),x'(t_k))] - t\sum_{k=1}^m[I_k(x(t_k)) + (1-t_k)\bar{I}_k(x(t_k),x'(t_k))] \\ &\quad + \frac{1}{1-\int_0^1 v(s)\mathrm{d}s}\int_0^1\left[\sum_{0<t_k<s}[I_k(x(t_k)) + (s-t_k)\bar{I}_k(x(t_k),x'(t_k))]v(s)\mathrm{d}s\right] \\ &\quad - \frac{\int_0^1 sv(s)\mathrm{d}s}{1-\int_0^1 v(s)\mathrm{d}s}\sum_{k=1}^m[I_k(x(t_k)) + (1-t_k)\bar{I}_k(x(t_k),x'(t_k))], \end{aligned} \tag{2.20}$$

where $H(t,s)$ is defined by (2.6).

Conversely, if $x \in PC^1[J,E]$ is a solution of (2.14). Evidently,

$$\Delta x|_{t=t_k} = I_k(x(t_k)), \quad (k = 1, 2, \dots, m).$$

Direct differentiation of (2.14) implies, for $t \neq t_k$

$$\begin{aligned} x'(t) &= (Tx)'(t) = -\int_0^t sw(s)f(s,x(s),x'(s),(Ax)(s),(Bx)(s))\mathrm{d}s \\ &\quad + \int_t^1 (1-s)w(s)f(s,x(s),x'(s),(Ax)(s),(Bx)(s))\mathrm{d}s \\ &\quad - \sum_{k=1}^m[I_k(x(t_k)) + (1-t_k)\bar{I}_k(x(t_k),x'(t_k))] + \sum_{0<t_k<t}\bar{I}_k(x(t_k),x'(t_k)), \end{aligned} \tag{2.21}$$

and

$$x'' = -w(s)f(s,x(s),x'(s),(Ax)(s),(Bx)(s)). \tag{2.22}$$

So $x \in C^2[J',E]$ and $\Delta x'|_{t=t_k} = \bar{I}_k(x(t_k),x'(t_k))$, $(k = 1, 2, \dots, m)$, and it is easy to verify that $x(0) = x(1) = \int_0^1 v(s)x(s)\mathrm{d}s$. The proof is complete. $\diamond$

From (2.21), we get

$$H_t'(t,s) = G_t'(t,s) = \begin{cases} 1-s, & 0 \le t \le s \le 1 \\ -s, & 0 \le s \le t \le 1. \end{cases} \tag{2.23}$$

For $S \subset PC^1[J,E]$, we denote $S' = \{x' : x \in S\} \subset PC[J,E]$, $S(t) = \{x(t) : x \in S\} \subset E$ and $S'(t) = \{x'(t) : x \in S\} \subset E (t \in J)$.

Lemma 2.2. *If $S \subset PC^1[J, E]$ is bounded and the elements of S' are equicontinuous on each J_k $(k = 1, 2, \dots, m)$, then*

$$\alpha_{PC^1}(S) = \max\{\sup_{t\in J} \alpha(S(t)), \sup_{t\in J} \alpha(S'(t))\}.$$

The proof of Lemma 2.2 is similar to that of Lemma 4.3.11 *in* [4].

Lemma 2.3. *Assume that* (H_1) *and* (H_2) *hold. Then, operator T is a strict set contraction from $B_r^{(1)}$ into $PC^1[J, E]$, where $B_r^{(1)} = \{x \in PC^1[J, E] : \|x\|_1 \le r\}$.*

Proof. By (H_1), it is clear that $T : B_r^{(1)} \to PC^1[J, E]$ is continuous and bounded. Let $S \subset B_r^{(1)}$ be arbitrarily given. So $T(S) \subset PC^1[J, E]$ is bounded.

By (2.21), we have, for $x \in PC^1[J, E]$, $t \ne t_k$ $(k = 1, 2, , \dots, m)$,

$$(Tx)'(t) = \int_0^1 H_t'(t, s)w(s)f(s, x(s), x'(s), (Ax)(s), (Bx)(s))ds - \sum_{k=1}^{m}[I_k(x(t_k)) + (1 - t_k)\bar{I}_k(x(t_k), x'(t_k))] + \sum_{0<t_k<t} \bar{I}_k(x(t_k), x'(t_k)), \tag{2.24}$$

where $H_t'(t, s)$ is given by (2.23). It is easy to see from (2.21) that the elements of $(T(S))'$ are equicontinuous on each J_k $(k = 1, 2, \dots, m)$, hence, Lemma 2.2 implies

$$\alpha_{PC^1} = \max\{\sup_{t\in J} \alpha(T(S))(t), \sup_{t\in J} \alpha(T(S))'(t)\}. \tag{2.25}$$

Using (2.5), (2.1)–(2.3) and arguing similarly as in the proof of Theorem 2.1.1 in [4], we get

$$\begin{aligned}
\alpha((T(S))(t)) \le{} & \frac{1}{4}\gamma \times \max_{s\in J} w(s) \times [c_1\alpha(S(J)) + c_2\alpha(S'(J)) + c_3 g_0\alpha(S(J)) + c_4 h_0\alpha(S(J))] \\
& + \sum_{0<t_k<t} [\alpha(I_k(S(t_k))) + (1 - t_k)\alpha(\bar{I}_k(S(t_k)), (S'(t_k)))] \\
& + \sum_{k=1}^{m}[\alpha(I_k((S)(t_k))) + (1 - t_k)\alpha(\bar{I}_k(S(t_k)), (S'(t_k)))] \\
& + \frac{\mu}{1-\mu} \sum_{0<t_k<s} [\alpha(I_k(S(t_k)) + (1 - t_k)\alpha(\bar{I}_k(x(t_k), x'(t_k))))] \\
& + \frac{1}{1-\mu} \sum_{k=1}^{m}[\alpha(I_k(S(t_k)) + (1 - t_k)\alpha(\bar{I}_k(x(t_k), x'(t_k))))] \\
\le{} & \frac{1}{4}\gamma \times \max_{s\in J} w(s) \times [(c_1 + c_3 g_0 + c_4 h_0)\alpha(S(J)) + c_2\alpha(S'(J))] \\
& + \sum_{0<t_k<t} [d_k\alpha(S(t_k)) + (1 - t_k)(\bar{d}_k\alpha(S(t_k)) + \hat{d}_k\alpha(S'(t_k)))] \\
& + \sum_{k=1}^{m}[d_k\alpha(S(t_k)) + (1 - t_k)(\bar{d}_k\alpha(S(t_k)) + \hat{d}_k\alpha(S'(t_k)))] \\
& + \frac{\mu}{1-\mu} \sum_{0<t_k<s} [d_k\alpha(S(t_k)) + (1 - t_k)(\bar{d}_k\alpha(S(t_k)) + \hat{d}_k\alpha(S'(t_k)))] \\
& + \frac{1}{1-\mu} \sum_{k=1}^{m}[d_k\alpha(S(t_k)) + (1 - t_k)(\bar{d}_k\alpha(S(t_k)) + \hat{d}_k\alpha(S'(t_k)))]
\end{aligned} \tag{2.26}$$

and the definition of $\alpha_{PC^1}(S)$ in Lemma 2 implies

$$\alpha(S(J)) \le 2\alpha_{PC^1}(S), \qquad \alpha(S'(J)) \le 2\alpha_{PC^1}(S), \tag{2.27}$$

where

$$S(J) = \{x(t) : x \in S, t \in J\}, \quad \text{and} \quad S'(J) = \{x'(t) : x \in S, t \in J\}.$$

On the other hand, similar to the proof of Lemma 2 in [25], we can obtain

$$\alpha(S(t_k)) \le \alpha_{PC^1}(S), \qquad \alpha(S'(t_k)) \le \alpha_{PC^1}(S), \quad k = 1, 2, \dots, m. \tag{2.28}$$

It follows from (2.26)–(2.28) that

$$\alpha(T(S))(t) \leq \bar{\Gamma}_r \alpha_{PC^1}(S), \quad \forall t \in J. \tag{2.29}$$

As the same way, by virtue of (2.20), (2.1)–(2.3), (2.27) and (2.28), we get

$$\alpha(T(S))'(t) \leq \hat{\Gamma}_r \alpha_{PC^1}(S), \quad \forall t \in J. \tag{2.30}$$

Finally, (2.26), (2.29) and (2.30) imply that

$$\alpha_{PC^1}(T(S)) \leq \Gamma_r \alpha_{PC^1}(S).$$

Noticing $\Gamma_r = \max\{\bar{\Gamma}_r, \hat{\Gamma}_r\} < 1$, we claim that T is a strict set contraction, and the Lemma is proved. ◇

3. Main results

In this section, we apply Lemma 1.1 to establish the existence of solutions for problem (1.1). Let us begin by introducing some notation. Define

$$\overline{\lim_{\sum_{i=1}^{4}\|u_i\|\to\infty}} \left(\sup_{t\in J} \frac{\|f(t, u_1, u_2, u_3, u_4)\|}{\sum_{i=1}^{4}\|u_i\|} \right) = \Lambda,$$

$$\overline{\lim_{\|u_1\|\to\infty}} \frac{\|I_k(u_1)\|}{\|u_1\|} = \Lambda_k \ (k = 1, 2, \ldots, m),$$

$$\overline{\lim_{\|u_1+u_2\|\to\infty}} \frac{\|\bar{I}_k(u_1, u_2)\|}{\|u_1\| + \|u_2\|} = \bar{\Lambda}_k \ (k = 1, 2, \ldots, m).$$

Theorem 3.1. *Let conditions* (H_1)–(H_3) *be satisfied. Suppose further that*

$$\delta = \max\{\delta_1, \delta_2\} < 1, \tag{3.1}$$

where

$$\delta_1 = \frac{1}{4}\gamma\Lambda(2 + g_0 + h_0)\int_0^1 w(s)\mathrm{d}s + \frac{3-\mu}{1-\mu}\sum_{k=1}^{m}[\Lambda_k + 2(1-t_k)\bar{\Lambda}_k]$$

and

$$\delta_2 = \gamma\Lambda(2 + g_0 + h_0)\int_0^1 w(s)\mathrm{d}s + \sum_{k=1}^{m}[\Lambda_k + 2(2-t_k)\bar{\Lambda}_k].$$

Then, problem (1.1) has at least one solution $x \in PC^1[J, E] \cap C^2[J', E]$.

Proof. By Lemma 2.3, operator T defined by (2.5) is a strict set contraction from $B_r^{(1)}$ into $PC^1[J, E]$, and by Lemma 2.1, we need only to show that T has one fixed points $x(t) \in PC^1[J, E] \cap C^2[J', E]$.

On account of (3.1), we can choose $\Lambda' > \Lambda$, $\Lambda'_k > \Lambda_k$ and $\bar{\Lambda}'_k > \bar{\Lambda}_k$ $(k = 1, 2, \ldots, m)$ such that

$$\delta'_1 = \frac{1}{4}\gamma\Lambda'(2 + g_0 + h_0)\int_0^1 w(s)\mathrm{d}s + \frac{3-\mu}{1-\mu}\sum_{k=1}^{m}[\Lambda'_k + 2(1-t_k)\bar{\Lambda}'_k] < 1, \tag{3.2}$$

and

$$\delta'_2 = \gamma\Lambda'(2 + g_0 + h_0)\int_0^1 w(s)\mathrm{d}s + \sum_{k=1}^{m}[\Lambda'_k + 2(2-t_k)\bar{\Lambda}'_k] < 1. \tag{3.3}$$

By the definition of Λ, there exists $l > 0$ such that

$$\|f(t, u_1, u_2, u_3, u_4)\| < \Lambda'\sum_{i=1}^{4}\|u_i\|, \quad \forall t \in J, \ u_i \in E, \ \sum_{i=1}^{4}\|u_i\| > l,$$

so

$$\|f(t, u_1, u_2, u_3, u_4)\| < \Lambda'\sum_{i=1}^{4}\|u_i\| + M, \quad \forall t \in J, \ u_i \in E, i = 1, 2, 3, 4, \tag{3.4}$$

where

$$M = \sup\left\{\|f(t,u_1,u_2,u_3,u_4)\| : t \in J,\ \sum_{i=1}^{4}\|u_i\| \le l\right\} < \infty.$$

Similarly, we have

$$\|I_k(u_1)\| < \Lambda'_k\|u_1\| + M_k, \quad \forall u_1 \in E\ (k = 1, 2, \ldots, m) \tag{3.5}$$

and

$$\|\bar{I}_k(u_1,u_2)\| \le \bar{\Lambda}'_k(\|u_1\| + \|u_2\|) + \bar{M}_k, \quad \forall u_1, u_2 \in E\ (k = 1, 2, \ldots, m), \tag{3.6}$$

where $M_k, \bar{M}_k$ are positive constants. Now, (2.5) and (3.4)–(3.6) imply

$$\begin{aligned}
\|(Tx)(t)\| &\le \frac{1}{4}\gamma\int_0^1 w(s)[\Lambda'(\|x(s)\| + \|x'(s)\| + \|(Ax)(s)\| + \|(Bx)(s)\|) + M]\mathrm{d}s \\
&\quad + \sum_{0<t_k<t}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\quad + \sum_{k=1}^{m}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\quad + \frac{1}{1-\mu}\sum_{0<t_k<s}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\quad + \frac{\mu}{1-\mu}\sum_{0<t_k<s}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\le \frac{1}{4}\gamma\int_0^1 w(s)[\Lambda'(\|x(s)\| + \|x'(s)\| + \|(Ax)(s)\| + \|(Bx)(s)\|) + M]\mathrm{d}s \\
&\quad + \sum_{k=1}^{m}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\quad + \sum_{k=1}^{m}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\quad + \frac{1}{1-\mu}\sum_{k=1}^{m}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\quad + \frac{\mu}{1-\mu}\sum_{k=1}^{m}\{(\Lambda'_k\|x(t_k)\| + M_k) + (1-t_k)[\Lambda'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\} \\
&\le \frac{1}{4}\gamma[\Lambda'(2 + g_0 + h_0)\|x\|_1 + M]\int_0^1 w(s)\mathrm{d}s \\
&\quad + \frac{3-\mu}{1-\mu}\sum_{k=1}^{m}[\Lambda'_k\|x\|_1 + M_k + 2(1-t_k)\bar{\Lambda}'_k\|x\|_1 + (1-t_k)\bar{M}_k] \\
&= \left\{\frac{1}{4}\gamma\Lambda'(2 + g_0 + h_0)\int_0^1 w(s)\mathrm{d}s + \frac{3-\mu}{1-\mu}\sum_{k=1}^{m}[\Lambda'_k + 2(1-t_k)\bar{\Lambda}'_k]\right\}\|x\|_1 \\
&\quad + \frac{1}{4}\gamma M\int_0^1 w(s)\mathrm{d}s + \frac{3-\mu}{1-\mu}\sum_{k=1}^{m}[M'_k + (1-t_k)\bar{M}'_k] \\
&= \delta'_1\|x\|_1 + M^{(1)},
\end{aligned} \tag{3.7}$$

where δ'_1 is defined by (3.2) and M^1 is defined by

$$M^{(1)} = \frac{1}{4}\gamma M\int_0^1 w(s)\mathrm{d}s + \frac{3-\mu}{1-\mu}\sum_{k=1}^{m}[M'_k + (1-t_k)\bar{M}'_k].$$

Similarly, from (2.24) and (3.2)–(3.5), we can get

$$\|(Tx)'(t)\| \le \delta'_2\|x\|_1 + M^{(2)}, \quad \forall t \in J, \tag{3.8}$$

where δ_2' is defined by (3.3) and M^2 is given by

$$M^{(2)} = M\int_0^1 w(s)\mathrm{d}s + \sum_{k=1}^{m}[M_k + (2-t_k)\bar{M}_k].$$

It follows from (3.7) and (3.8) that

$$\|Tx\|_1 \le \delta'\|x\|_1 + M', \quad \forall x \in PC^1[J, E],$$

where

$$\delta' = \max\{\delta_1', \delta_2'\} < 1, \quad M' = \max\{M^{(1)}, M^{(2)}\}.$$

Hence, we can choose a sufficiently large $r > 0$ such that $T(B_r^{(1)}) \subset B_r^{(1)}$.

On the other hand, by Lemma 2.3, T is a strict set contraction from $B_r^{(1)}$ into $B_r^{(1)}$. Consequently, Lemma 1.1 implies that T has a fixed point in $B_r^{(1)}$, and the proof is complete. ◇

Remark 3.1. Condition (3.1) is certainly satisfied if $\|f(t,u_1,u_2,u_3,u_4)\|/\sum_{k=1}^m\|u_i\| \to 0$ uniformly in $t \in J$ as $\sum_{k=1}^m\|u_i\| \to \infty$, $\|I_k(u_1)\|/\|u_1\| \to 0$ as $\|u_1\| \to \infty$ and $\|I_k(u_1,u_2)\|/(\|u_1\| + \|u_2\|) \to 0$ as $(\|u_1\| + \|u_2\|) \to \infty$ $(k = 1, 2, \ldots, m)$.

To illustrate how our main results can be used in practice we present an example.

Example 3.1. Consider the following boundary value problem of finite system of scalar second-order impulsive integro-differential equation

$$\begin{cases} -x_i'' = t^{\frac{1}{2}}\left\{\sqrt[3]{t - x_i + x_{i+1}'} - \dfrac{1}{20}x_{i+2}' - 3\ln(1 + x_{2i}^2)\right. \\ \quad \left. -\dfrac{1}{3}\left[\left(\displaystyle\int_0^t \mathrm{e}^{-ts}x_{i+2}(s)\mathrm{d}s\right)^2 + \left(\displaystyle\int_0^1 cos(t-s)x_{2i}(s)\mathrm{d}s\right)^2\right]^{\frac{1}{5}}\right\}, \quad t \in J, t \ne \dfrac{1}{2}, \\ \Delta x|_{t_1=\frac{1}{2}} = \dfrac{1}{10}x_{i+1}\left(\dfrac{1}{2}\right), \\ \Delta x'|_{t_1=\frac{1}{2}} = \dfrac{1}{6}\left(x_i\left(\dfrac{1}{2}\right) - x_{i+1}'\left(\dfrac{1}{2}\right)\right), \\ x_i(0) = x_i(1) = 0, \end{cases} \tag{3.9}$$

where $x_{n+i} = x_i, x_{n+i}' = x_i'$ $(i = 1, 2\ldots, n)$.

Conclusion. Problem (3.9) has at least one solution.

Proof. Let $E = R^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in R, i = 1, 2, \ldots, n\}$ with the norm $\|x\| = \max_{1\le i\le n}|x_i|$, system (3.9) can be translated into the form of (1.1) in E. In this situation, $g(t,s) = \mathrm{e}^{-ts}$, $h(t,s) = \cos(t-s)$, $u_i = (u_{i1}, u_{i2}, \ldots, u_{in})$ $(i = 1, 2, 3, 4)$, $f = (f_1, f_2, \ldots, f_n)$, in which

$$\begin{aligned} f_i(t,u_1,u_2,u_3,u_4) &= \sqrt[3]{t - x_i + x_{i+1}'} - \frac{1}{20}x_{i+2}' - 3\ln(1 + x_{2i}^2), \\ &- \frac{1}{3}\left[\left(\int_0^t \mathrm{e}^{-ts}x_{i+2}(s)\mathrm{d}s\right)^2 + \left(\int_0^1 \cos(t-s)x_{2i}(s)\mathrm{d}s\right)^2\right]^{\frac{1}{5}}, \end{aligned} \tag{3.10}$$

and $m = 1$, $t_1 = \frac{1}{2}$, $I_1 = (I_{11}, I_{12}, \ldots, I_{1n})$, $\bar{I}_1 = (\bar{I}_{11}, \bar{I}_{12}, \ldots, \bar{I}_{1n})$ with

$$I_{1i}(u_1) = \frac{1}{10}u_{1i+1}, \qquad \bar{I}_{1i}(u_1,u_2) = \frac{1}{6}(u_{1i} - u_{2i+1}) \tag{3.11}$$

and $w(t) = t^{\frac{1}{2}}$, $v(t) \equiv 0$, $\forall t \in J$.

From the definition of u_i and f_i, we can obtain that $f \in C(J \times R^n \times R^n \times R^n \times R^n, R^n)$. Similarly, $I_1 \in C(R^n, R^n)$, $\bar{I}_1 \in C(R^n \times R^n, R^n)$, and for any $r > 0$, f is bounded and uniformly continuous on $J \times B_r \times B_r \times B_r \times B_r$. In the same way as in Example 3.2.1 in [4], we can show that (2.1) is satisfied for $c_i = 0 (i = 1, 2, 3, 4)$. By (3.11), it is not difficult to prove that (2.2) and (2.3) are satisfied for $d_1 = \frac{1}{10}$, $\bar{d}_1 = \hat{d}_1 = \frac{1}{6}$. Hence, (2.4) is also satisfied since $\mu = 0$, $\gamma = 1$, $g_0 = 1$, $h_0 = 1$, $\Gamma_r = \frac{4}{5}$ and $\bar{\Gamma}_r = \frac{3}{20}$.

Since $v(t) \equiv 0$, $\forall t \in J$, we get $H(t,s) = G(t,s)$.

On the other hand, from (3.10) and (3.11) we have

$$\|f(t,u_1,u_2,u_3,u_4)\| \le \sqrt[3]{t+\|u_1\|+\|u_2\|}+\frac{1}{20}\|u_2\|+3\ln(1+\|u_1\|^2)+\frac{1}{3}(\|u_3\|^2+\|u_4\|^2)^{\frac{1}{5}},$$

and

$$\|I_1(u_1)\| = \frac{1}{10}\|u_1\|, \qquad \|\bar{I}_1(u_1,u_2)\| = \frac{1}{6}(\|u_1+u_2\|),$$

so $\Gamma \le \frac{1}{6}$, $\Gamma_1 \le \frac{1}{10}$, $\bar{\Gamma}_1 \le \frac{1}{6}$, and therefore (3.1) is satisfied because $\delta_1 = \frac{5}{6}$ and $\delta_2 = \frac{11}{15}$. Thus, our conclusion follows from Theorem 3.1. $\diamond$

Acknowledgement

The authors thank the referee for his/her valuable suggestions. These have greatly improved this paper.

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